functions of several variables

- How can we tell what they look like?
- What do partial derivatives tell us?

Graph of a function on $\mathbb{R}^2$ is the surface $z = f(x,y)$

$$f : \underset{\text{domain}}{\mathbb{R}^2} \longrightarrow \mathbb{R}$$

we plot $(x,y,z) = (x,y,f(x,y))$

<u>Example</u> $z = x^2 + y^2 + 2y$

At each point (x,y) in $\mathbb{R}^2$, the vertical position is given by $z = x^2 + y^2 + 2y$.

$$f(x,y) = x^2 + y^2 + 2y.$$

The partial derivatives are:

$$f_x = D_x f = D_1 f = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Our derivative formulas are the same for these fcns. We just have to remember that the other variables are constants.

$$f(x,y) = x^2+y^2+2y$$

$$f_x = 2x$$ ← instantaneous rate of change of $f(x,y)$ as x increases, at (x,y).

Can also imagine this as the slope of the tangent line at (x,y,z), if you slice the graph where y is constant.

eg. at $(1,-2,1)$, the surface goes
$1^2+(-2)^2+2(-2)=1$
up with a slope of $2(1)=2$ as x increases.

$$f(x,y) = x^2+y^2+2y$$

$$f_y = 2y+2$$

↑ at $(1,-2,1)$, the slope in the y-direction of the surface is $2(-2)+2=-2$ as y increases.

More examples of partial derivatives.

① $xy\,e^{xy^2} = g(x,y)$.

Find g_x, $\frac{\partial g}{\partial y}$.

$$\left[xye^{xy^2}\right]_x \overset{\text{product}}{=} (xy)\left(e^{xy^2}\right)_x + (xy)_x e^{xy^2}$$

$$\underset{\text{chain}}{=} xy\cdot y^2e^{xy^2} + ye^{xy^2}$$

$$\left[xye^{xy^2}\right]_y = (xy)\left(e^{xy^2}\right)_y + (xy)_y e^{xy^2}$$

$$= xy\left(2xye^{xy^2}\right) + xe^{xy^2}$$

$$= 2x^2y^2e^{xy^2} + xe^{xy^2}.$$

(2) $h(x,y) = \cos(xy^2)$: Find $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$

(3) $A(x,y,z) = xy + \frac{1}{xz}$

Find A_x, A_y, A_z.

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(2) $h(x,y) = \cos(xy^2)$

$h_y = -2xy\sin(xy^2)$

$h_x = -y^2\sin(xy^2)$

(3) $A(x,y,z) = xy + \frac{1}{xz} = xy + x^{-1}z^{-1}$

$A_x = y - \frac{1}{zx^2}$

$A_y = x$

$A_z = \frac{-1}{xz^2}$

New technique of understanding surface graphs $z = f(x,y)$.

Contour Diagram:

Graph $z = \text{constant}$ in the xy plane for several different constants.

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eg. $f(x,y) = x^2 + y^2 + 2y$.

$z = -1$

$$-1 = x^2 + y^2 + 2y$$

$$0 = x^2 + y^2 + 2y + 1$$

$$0 = x^2 + (y+1)^2$$

$\iff x = 0$ and $y + 1 = 0$

$y = -1$

the point $(0,-1)$

$z = 0$

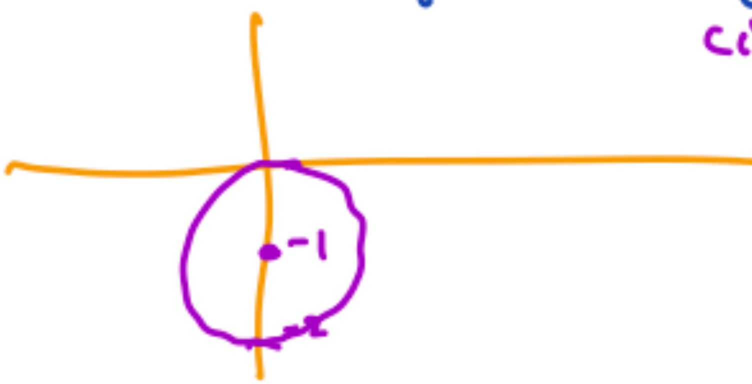
$0 = x^2 + y^2 + 2y$

add 1 to both sides

$1 = x^2 + y^2 + 2y + 1$

$1 = x^2 + (y+1)^2$

circle centered at $(0,-1)$ of radius 1



$z = 1$

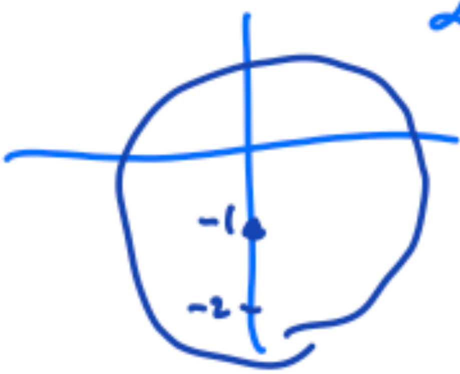
$1 = x^2 + y^2 + 2y$

add 1

$2 = x^2 + y^2 + 2y + 1$

$2 = x^2 + (y+1)^2$

circle centered at $(0,-1)$ of radius $\sqrt{2}$



$z = x^2 + y^2 + 2y$

$z + 1 = x^2 + y^2 + 2y + 1$

$(\sqrt{z+1})^2 = x^2 + (y+1)^2$

Circle of radius $\sqrt{z+1}$ centered at $(0,-1)$